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Galerkin approximations to static and dynamic localization problems

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Abstract

A new Galerkin-type procedure is established which, unlike the classical approach, does not rely on the final shape being composed of linearly independent modes. The procedure is applied to the evolution of a localized buckle of a thin elastic strip within a visco-elastic medium. Unlike the related elastic problem, no clear-cut linear eigenvalues exist to model wavelength and exponential growth/decay in the tails of the buckle pattern. The new procedure introduces variables to measure these effects, and allows them to change in time. This results in a more natural evolutionary process than with fixed mode shapes. Analysis is run within an algebraic manipulator (MAPLE) and checked against that of a numerical boundary-value solver (COLPAR).

1. Introduction

In parallel with developments in other areas of mechanics and applied mathematics (Champneys et al., 1997), the analysis of buckle pattern localization is fast emerging as an important area of study in structural mechanics. Special techniques are often required, tuned specifically to describing the inherently nonlinear and non-unique responses that result. Methods range from analytical (e.g. perturbation) methods, through part numerical and part analytic (Wadee et al., 1997), to fully numerical methods (Champneys and Toland, 1993). Typical applications are to embedded strut, plate and shell problems, with emerging interest in the field of structural geology.

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Formulations for such problems must involve at least one independent spatial variable x, which can sometimes usefully be interpreted in a dynamical sense as time t (Hunt et al., 1989), whereupon the system appears as reversible and usually hamiltonian. In the modelling of geological systems, however, real time also enters the frame, and underlying governing equations become partial differential (PDEs) in space and time, rather than ordinary differential (ODEs) just in space. Analytical equipment that is useful for the latter is not necessarily available in the context of the former. In particular, in the archetypal case of an embedded strut, linear eigenvalue information suggests a modulated periodicity that defines both wavelength and exponential growth/decay in the tails of a localized solution (henceforth collectively referred to as *shape factors*) (Wadee et al., 1997). In the geological setting, such information is not readily available from the linearized PDE.

Available analytical techniques are also liable to be more restricted in the case of PDEs. Variational approaches such as the Rayleigh–Ritz method, which might work well for an elastic system, have no immediate counterparts in the corresponding dissipative, viscous or visco-elastic, context of geology. Galerkin procedures, based on underlying differential equations, face less restriction, but by comparison also appear less flexible.

As an example, we compare here two models, one describing an elastic strut supported by a nonlinear elastic (Winkler) medium and leading to an ODE, and the other in which the support is replaced by a nonlinear visco-elastic medium and leading to a PDE. For the first problem, changing the load alters the form of the buckle pattern; we compare two Galerkin procedures, one in which the linear eigenvalue information is utilized directly, and one for which the shape factors are introduced as free variables and allowed to choose themselves via the minimization of excess energy. The latter involves an extension of the Galerkin principle to modes which are no longer linearly independent, and gives results that agree directly with those of a recent Rayleigh–Ritz formulation of the same problem (Wadee et al., 1997). For the visco-elastic problem, the time dimension again leads in general to buckle patterns that are liable to alter shape. Now however no eigenvalue information exists to guide the choice of trial functions; wavelengths and growth/decay can either be treated as arbitrarily constant, or as with the elastic problem be given the status of variables in the new Galerkin procedure. In this last example there is no Rayleigh–Ritz counterpart for comparison. In all cases we compare against numerically-obtained solutions found using the boundary value solver COLPAR.

For systems that display localized shapes, Galerkin and related procedures such as are developed here have clear potential. Unlike asymptotic double-scale results, they are capable of representing the deflected shape over entire load ranges, well away from any point of expansion for example. This is particularly useful for modelling geological systems, where forces due to tectonic compression dissipate over time. In this context, the ability of a Galerkin model to discover for itself the wavelength and growth/decay characteristics in the tails of a modulated harmonic function is undoubtedly a useful feature.

2. Static model: the ODE

To develop and test the performance of a Galerkin method suitable for the study of buckle pattern localization, we start with the derivation of a well known static model (Potier-Ferry, 1983). Comparisons will be with independently obtained numerical solutions.

2.1. Derivation of the formulation from a variational principle

Consider the total potential energy functional, V, of a strut with bending stiffness EI of indefinite length lying on an elastic (Winkler) foundation with a softening cubic nonlinearity with restoring force per unit length $F = ky - cy^3$ where k and c are positive constants. The strut has a compressive axial force acting on it which maintains its magnitude and direction throughout all deformations but may be parametrically varied (Fig. 1). After nondimensionalizing (i.e. putting $EI = k = c = 1$), and considering only first-order bending and work-done-by-load terms, the total potential energy is (Thompson and Hunt, 1973)

$$V = \int_{-\infty}^{\infty} \left(\frac{1}{2} y''^2 - \frac{1}{2} P y'^2 + \frac{1}{2} y^2 - \frac{1}{4} y^4 \right) dx \tag{1}$$

where a prime indicates differentiation with respect to x. The first term is the strain energy of bending, the second is the work done by the load P and the remainder are energy stored in the elastic foundation. The integrand can be treated as a spatial Lagrangian, $\mathscr{L}(y, y', y'')$, of the system. Equilibrium is given by stationary values of the integral. Performing the calculus of variations on (1) gives (Wadee et al., 1997)

$$\delta V = \left[\frac{\partial \mathscr{L}}{\partial y'} \delta y + \frac{\partial \mathscr{L}}{\delta y''} \delta y' - \frac{d}{dx} \left(\frac{\partial \mathscr{L}}{\partial y''} \right) \delta y \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \left\{ \frac{\partial \mathscr{L}}{\partial y} - \frac{d}{dx} \left(\frac{\partial \mathscr{L}}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial \mathscr{L}}{\partial y''} \right) \right\} \delta y \, dx = 0, \tag{2}$$

the integrand of which contains the Euler–Lagrange equation for the system, in this case given by

$$y'''' + P y'' + y - y^3 = 0. \tag{3}$$

The term in square brackets accounts for conditions at the boundaries and disappears when we consider localized solutions the envelopes of which decay exponentially for large $|x|$. We are left with

$$\int_{-\infty}^{\infty} \left\{ \frac{\partial \mathscr{L}}{\partial y} - \frac{d}{dx} \left(\frac{\partial \mathscr{L}}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial \mathscr{L}}{\partial y''} \right) \right\} \delta y \, dx = 0. \tag{4}$$

Eq. (3) has a critical point at $P^C = 2$ where the flat fundamental state loses stability at an unstable-symmetric point of bifurcation and encounters equilibrium paths corresponding to buckled states of the strut (Thompson and Hunt, 1973). Approximate solutions can be obtained using classical perturbation techniques, and can be either periodic or localized in form.

2.1.1. The traditional Galerkin method

The traditional Galerkin procedure (Fox, 1987) may be seen as being derived from Eq. (2) by assuming that the modes which go to make up y are given by

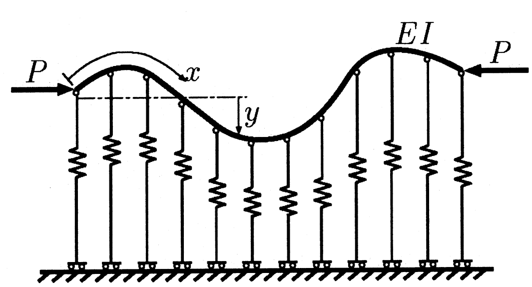


Fig. 1. A strut resting on an elastic foundation.

$$y = \sum_{i=1}^{n} A_i \phi_i(x) \tag{5}$$

where each A_i is an undetermined constant amplitude of each shape function, ϕ_i. If δy in (4) is replaced in turn by $(\partial y/\partial A_i)\delta A_i$ we get n integral equations of the form,

$$\int_{-\infty}^{\infty} \{y'''' + Py'' + y - y^3\}\phi_i \, dx = 0. \tag{6}$$

The traditional Galerkin method is thus recovered. The coefficient δA_i has been removed since it is arbitrary and we are only considering the first (weak) variation of V. Depending on the form of y in (5), we can perform periodic or localized buckling analyses of the strut. Whiting (1997) performed the latter using the following assumed form

$$y = A_1 \mathrm{sech}\alpha x \cos \beta x + A_2 \mathrm{sech}\alpha x \tanh \alpha x \sin \beta x + A_3 \mathrm{sech}^3\alpha x \cos \beta x + A_4 \mathrm{sech}^3\alpha x \tanh \alpha x \sin \beta x. \tag{7}$$

This expression was developed from a double-scale perturbation analysis (Wadee et al., 1997). The variables α and β are related to the degree of localization and wavelength of the deflection pattern. A good approximation is to assume that they are given by

$$\alpha = \sqrt{\frac{P^{\mathrm{C}}}{4} - \frac{P}{4}}$$

$$\beta = \sqrt{\frac{P^{\mathrm{C}}}{4} + \frac{P}{4}} \tag{8}$$

where $\pm\alpha \pm i\beta$ are the four eigenvalues of the linearized differential equation.

2.1.2. The modified Galerkin method

The above is shown by Whiting (1997) to give good results apart from a small range of P near the critical point P^{C}. This is despite the fact that the parameters α and β are functions of P only. A further development in a Rayleigh–Ritz context is to allow α and β to act as generalized coordinates as well as the A_is (Wadee et al., 1997). Even though the ‘modes’ are no longer linearly independent this can be directly translated into a Galerkin procedure; the integrand (4) still holds but expressions $(\partial y/\partial \alpha)\delta\alpha$ and $(\partial y/\partial \beta)\delta\beta$, namely

$$\delta\alpha \int_{-\infty}^{\infty} \{y'''' + Py'' + y - y^3\} \frac{\partial y}{\partial \alpha} \, dx = 0$$

$$\delta\beta \int_{-\infty}^{\infty} \{y'''' + Py'' + y - y^3\} \frac{\partial y}{\partial \beta} \, dx = 0, \tag{9}$$

now must be included. This leads to more complicated integrals which nevertheless can be evaluated in closed form by the technique of contour integration (Stephenson and Radmore, 1990). In the current study, use is made of the algebraic manipulation software MATHEMATICA (Wolfram Research Inc., 1995) and MAPLE (Waterloo Maple Inc., 1996). The resulting six nonlinear algebraic equations are then solved numerically using a six-dimensional Newton–Raphson technique (Press et al., 1992). The improvement over the traditional Galerkin procedure can be gauged by comparing the average (squared) residual defined by

$$R_{\mathrm{av}} = \sqrt{\int_{-\infty}^{\infty} R^2 \,\mathrm{d}x},$$

where

$$R = y'''' + Py'' + y - y^3. \tag{10}$$

Fig. 2 shows the favourable comparison of this new Galerkin procedure with the classical form over the entire post-buckling response. A corresponding improvement in the plot of load P to first-order end shortening, $\mathscr{E}$, defined by

$$\mathscr{E} = \frac{1}{2}\int_{-\infty}^{\infty} y'^{\,2} \,\mathrm{d}x, \tag{11}$$

is shown in Fig. 3 (c.f. Whiting, 1997). Here and henceforth we shall refer to the original Galerkin method as G1 and the modified method as G2.

A point worth emphasizing is that the conventional Galerkin analysis gives identical results to those of a localized Rayleigh–Ritz analysis when α and β are the same functions of P, while this enhanced analysis again repeats the Rayleigh–Ritz results with α and β being treated as generalized coordinates and allowed to choose themselves. Both forms of Rayleigh–Ritz modelling appear in Wadee et al. (1997). Variations of α and β for each are as shown in Fig. 4. As outlined in that paper, both Galerkin and Rayleigh–Ritz methods derive from the same variational principle with simply the order of differentiation and integration being swapped. The exact equivalence when the modes are not linearly independent is, to the best of our knowledge, reported here for the first time.

2.2. Comparison with independent numerical solutions

Figs. 2 and 3 show that the modified Galerkin method performs consistently better than the original method. To obtain numerical solutions to Eq. (3) we employed the collocation software COLNEW (Ascher et al., 1995). This is a boundary-value solver requiring appropriate boundary conditions. We utilize a known symmetry in the primary localized solution, associated with the underlying reversibility of the equation (Champneys, 1994), by taking one boundary ($x = 0$) to be at the centre of localization under the symmetric section condition $y' = y''' = 0$ (Hunt et al., 1989). Solutions are then obliged to be symmetric about the y-axis. The other boundary condition is taken to be the linearized stable manifold

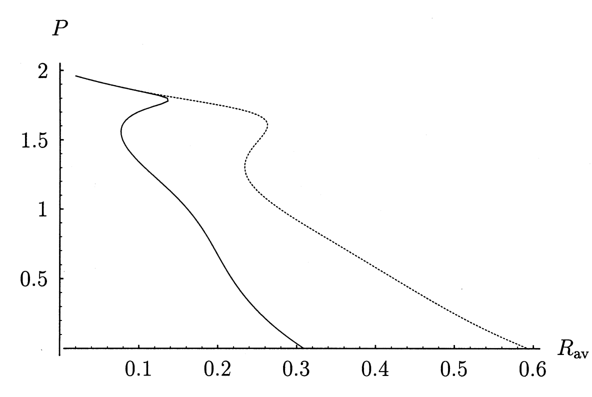


Fig. 2. Graphs of P vs R_{av} for the traditional (dashed line) and modified (solid line) Galerkin methods.

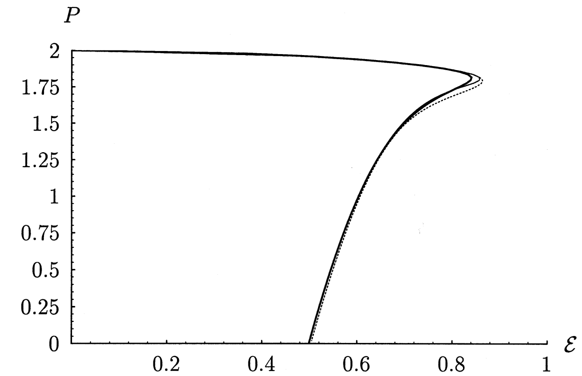


Fig. 3. End shortening for the elastic strut on an elastic foundation found using numerical and Galerkin approaches. Thick line: numerical solution. Dashed line: solution using G1. Thin line: solution using G2.

i.e. the solution to the linearized equation which converges on $y = 0$ as $x \to \infty$ (Hunt and Wadee, 1991). Whatever boundary condition is actually in place, this gives a good approximation to the localized response as deflection becomes small and decays towards the flat state. Favourable comparison of the deflections at three widely-ranging load values is shown in Fig. 5. Further evidence for the accuracy of the Galerkin results is shown in the graph of central deflection vs P for the entire post-buckling range (Fig. 6).

3. Dynamic model: the PDE

The Galerkin modifications of the previous section derive from a variational principle. The next problem to be examined using this method is that of a visco-elastic system in which the long-term buckling behaviour is triggered by an initial elastic phase followed by viscous dissipation in time (Hunt et al., 1996). This dynamic process models folding of geological strata under high temperature and pressure (Hunt et al., 1997).

No variational principle is available in the derivation of the PDE but the Galerkin formulation still

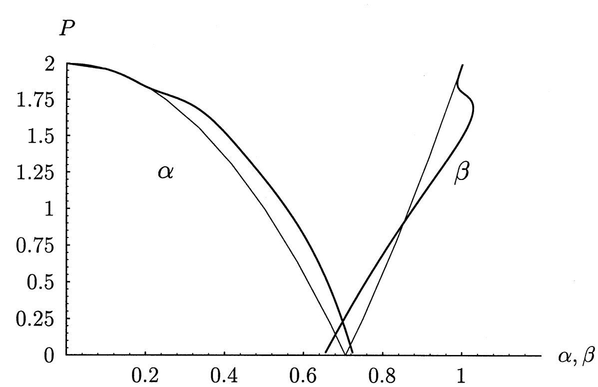


Fig. 4. Variation of α and β with load, P, for the strut on an elastic foundation (3). Thin line: values taken from the linearized eigenvalues (8). Thick line: values found by method G2.

gives improved results as shown below. The governing equation in $y(x,\ t)$, taken directly from Hunt et al. (1996)

$$\frac{\partial}{\partial t}\left(\frac{\partial^4 y}{\partial x^4}+P\frac{\partial^2 y}{\partial x^2}+y-y^3\right)+\frac{\partial^4 y}{\partial x^4}+P\frac{\partial^2 y}{\partial x^2}=0, \tag{12}$$

is appropriate for an elastic strut supported by a nonlinear Winkler foundation that comprises springs and dashpots in series. Depending on the prevailing loading conditions, the load parameter P can vary

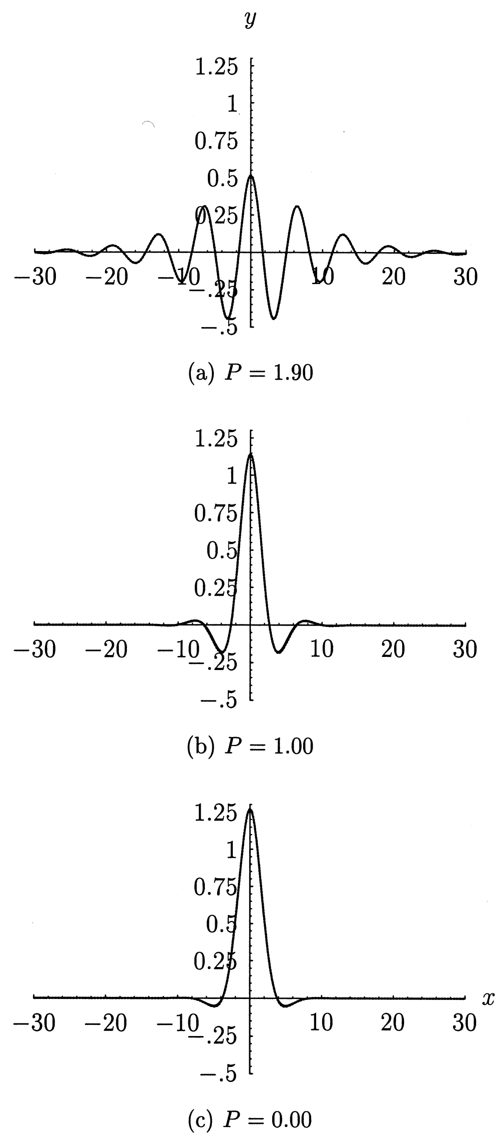


Fig. 5. Localized deflection profiles of Eq. (3) for the load values shown. The difference between numerical and Galerkin solutions cannot be discerned.

with time. In the current study, we shall assume that the system is held under constant end-shortening (rigid loading) ($\mathrm{d}\mathscr{E}/\mathrm{d}t = 0$). An alternative might be to assume that the rate of change of end shortening were constant, or indeed that $\mathrm{d}P/\mathrm{d}t = 0$ (dead loading). The point to note is that P and $\mathscr{E}$ are not independent.

A key difference between this model and the static one of the previous section is that the elastic case has localization characteristics (i.e. α and β) derivable from the linear eigenvalues of the governing ODE. Good results are obtained either by assuming that these are given by (8) or by adopting them as initial guesses and allowing α and β to vary as generalized coordinates using simple continuation. In the visco-elastic case no such guidance exists. It is true that the initial deflection is governed by the elastic properties of the strut and foundation but, with evolution in time, the energy-absorbing (viscous) properties of the foundation will affect the shape factors α and β. As the traditional Galerkin technique (G1) allows only the amplitudes of the mode shapes to vary, a method that allows α and β also to vary is highly desirable.

3.1. Solution procedure for the Galerkin methods

Method G2 developed above offers no improvement over the full Rayleigh–Ritz analysis of Wadee et al. (1997) for the elastic case but has the advantage that it can be applied to the non-conservative problems of visco-elasticity. To do this, the modal form of y in (7) is substituted into (12) and the left-hand side is treated as the residual. The generalized coordinates, A_i, α and β, are assumed to be functions of time. Each integral (4) is then evaluated in closed form for each $(\partial y/\partial A_1)\delta A_1$, $(\partial y/\partial A_2)\delta A_2$, ..., $(\partial y/\partial \beta)\delta\beta$ which gives six algebraic equations. We ensure that end shortening is held constant by insisting on the condition $\mathrm{d}\mathscr{E}/\mathrm{d}t = 0$. The seven equations thus formed are linear in $\mathrm{d}A_1/\mathrm{d}t$ etc. and $\mathrm{d}P/\mathrm{d}t$ and may be solved by matrix inversion. A finite-difference relationship is then used to find the values at the next time step, viz.

$$A_1(t + \Delta t) = A_1(t) + \left.\frac{\mathrm{d}A_1}{\mathrm{d}t}\right|_t \Delta t. \tag{13}$$

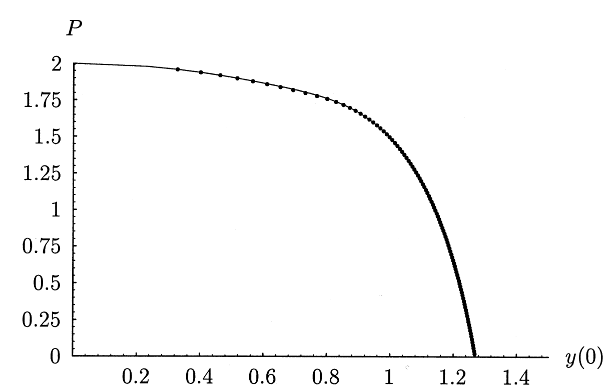


Fig. 6. Graphs of central deflection, $y(0)$, vs P for primary localized buckling solutions of (3). Continuous line: numerical solution. Discrete points: Galerkin method G2.

3.2. Discretization of time for numerical solutions

In order to solve (12) numerically using available software, it is transformed into a set of ODEs in space with the time dimension being discretized using a backward difference scheme with a step size of Δt. The process yields a set of ODEs in $y_n(x)$ where n denotes the deflection at time $n\Delta t$,

$$y_n'''' + P_n y_n'' + \frac{y_n - y_n^3}{1+\Delta t} = \frac{y_{n-1}'''' + P_{n-1}y_{n-1}'' + y_{n-1} - y_{n-1}^3}{1+\Delta t}. \tag{14}$$

The software used to solve this discretized system for constant end shortening was COLPAR (Bader and Kunkel, 1989)—a descendent of COLNEW that allows constraints to be included. The boundary conditions used to solve (14) were the same as for the elastic case, i.e. the symmetric section at $x = 0$ and the linearized stable manifold at $x \approx 100$.

The initial response, $y_0(x)$ is assumed to be that of an elastic strut precisely of the form given in section 2. The justification is that we are attempting to model folding of strata which is initiated by some disturbance that takes place instantaneously (in comparison with geological time scales). In all calculations, we have used $\Delta t = 0.01$ and $P_0 = 1.97$.

3.3. Linear PDE with localized triggering mode

The assumption of an initial localized mode is made on the grounds that the initiation of geological folding occurs quickly when compared with the subsequent process of viscous dissipation. Under conditions of constant end shortening, the deflection grows moderately compared with, say, growth under dead loading conditions ($\mathrm{d}P/\mathrm{d}t = 0$). It has recently been shown by Budd et al. (1999) that, even in the case of a purely viscous linear foundation, the imposition of the constant-end-shortening constraint (11) is sufficient to cause buckle patterns to localize; the constraint itself is enough of a nonlinearity to promote such behaviour. We therefore also examine the linear visco-elastic PDE (12),

$$\frac{\partial}{\partial t}\left(\frac{\partial^4 y}{\partial x^4} + P\frac{\partial^2 y}{\partial x^2} + y\right) + \frac{\partial^4 y}{\partial x^4} + P\frac{\partial^2 y}{\partial x^2} = 0. \tag{15}$$

It may be sensible to allow such a system to trigger in the form of a localized buckle by including an initial nonlinearity, but then allow the profile to evolve according to the linear equation.

3.4. Comparisons of the Galerkin formulations

3.4.1. Results for the nonlinear PDE

The plot of average residual vs time is shown in Fig. 7 for the two schemes and shows that, as for the elastic system, the residual for G2 is smaller than for the G1.

Comparisons of the deflected shape between the two Galerkin approaches and numerical solutions are shown in Figs. 8 and 9 for the values of t shown. For $t < 10$, results for both the weighted residual methods are close to the numerical solution but G2 performs better than G1 in the tails of the localization. The shape factors α and β chosen are dependent on the load at the initial elastic state of the system (in this case, $P(0) = 1.97$). As the load changes during evolution, so do the wavelength and rate of decay of the deflection pattern (see Figs. 8 and 9). Unlike for the ODE, it is not possible to calculate appropriate new values for the shape factors with change in load.

Method G2 performs consistently better but does not follow the numerical solution closely beyond about $t = 15$. A drift in the phase with x of the solutions is also apparent for larger t, although again

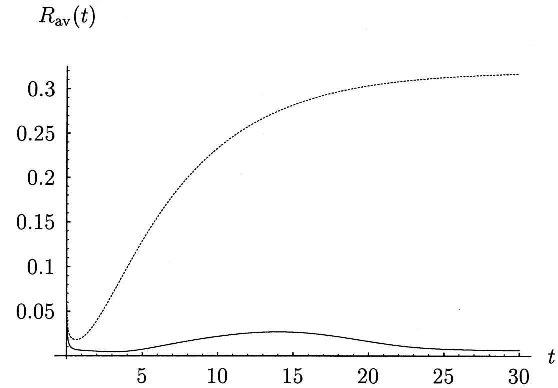


Fig. 7. Variation with time of the average residual for PDE (12). Dashed line: method G1. Solid line: method G2.

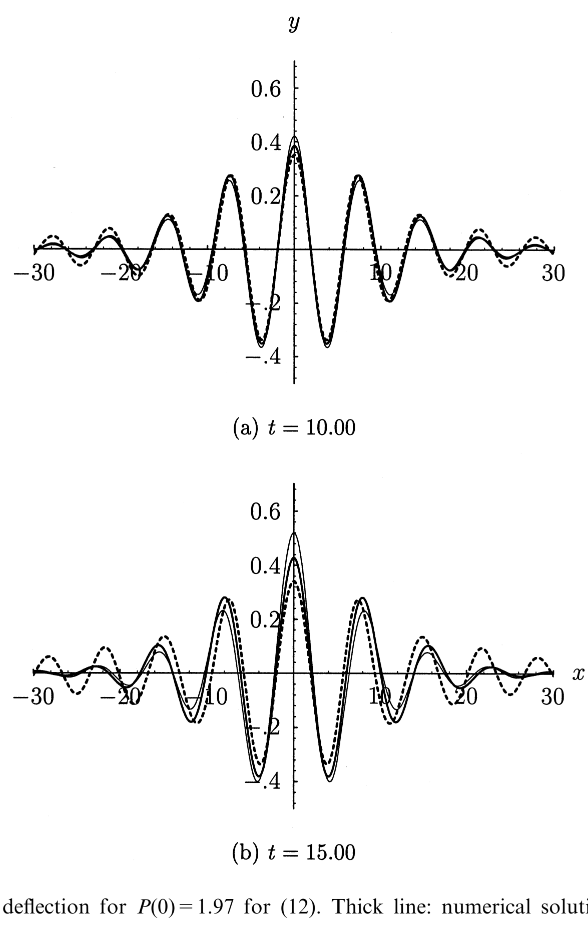
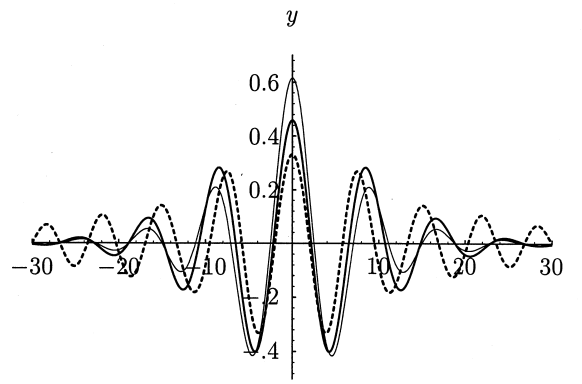
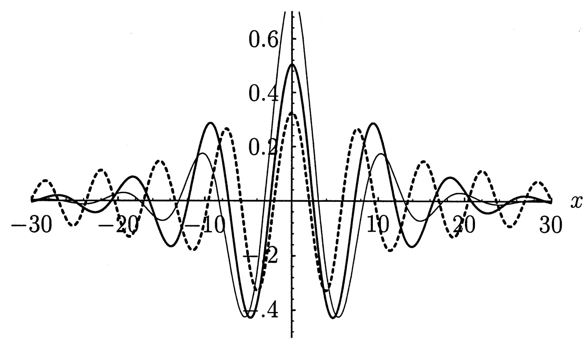


Fig. 8. Evolution of a localized deflection for $P(0) = 1.97$ for (12). Thick line: numerical solution. Thin line: solution using G2. Dashed line: solution using G1.



(a) $t = 20.00$



(b) $t = 30.00$

Fig. 9. Further evolution of the deflection pattern (continuation from Fig. 8).

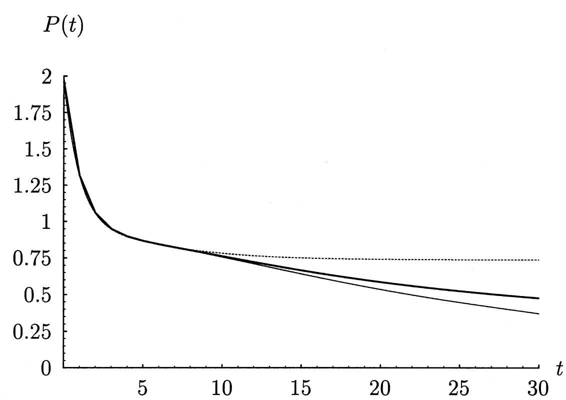


Fig. 10. Graph of change of load vs time for Eq. (12) under conditions of rigid loading. Thick line: numerical solution. Thin line: solution with method G2. Dashed line: solution with method G1.

the discrepancy is smaller for the newer scheme. Further evidence for the improved performance can be seen in the comparison of $P(t)$ vs t in Fig. 10 where graphs for the collocation solution and the newer Galerkin method only start to diverge significantly after about $t = 15$. Even after this time, although the comparisons are less good, the newer method does preserve some semblance of agreement with the numerical solution because the decay of amplitude is recalculated at each time step. Fig. 11 shows the variation of central deflection with time. According to the numerical solution, the deflection increases monotonically with time a feature that is reproduced in method G2. With G1, however, the central deflection starts to decrease after about $t = 5$. To achieve better accuracy in general for the PDE, more modes need to be used to approximate y. It is at present unclear which further modes would be appropriate, since the original choices were based on an elastic phase; there is no clear guarantee that extra such modes would be appropriate. Nevertheless, the ability to treat α and β as generalized coordinates is seen to provide significant extra freedom to the modelling process.

3.4.2. Results for the linear PDE

Eq. (15) is solved using the modified Galerkin method and using the collocation software. Comparisons are qualitatively similar to the nonlinear case. In particular, as in Fig. 11, the monotonic increase of the central deflection with time can be seen (Fig. 12). Both the numerical method and our extended Galerkin method pick this behaviour out whereas the original method predicts that this first rises and then falls. The load vs time graph (Fig. 13) shows that the Galerkin method remains close to the numerically-obtained results (Fig. 12). Fig. 14 shows the direct comparison of numerical and G2 solutions at $t = 20.00$ and $t = 30.00$.

4. Conclusions

Primary localized solutions, i.e. those whose amplitude envelope has a single peak, are a class of buckling solutions which are of practical and analytical importance in structural mechanics. In the elastic system, localized deflection is reminiscent of solitary-wave type solutions encountered in fluid mechanics (Champneys et al., 1997). Both Rayleigh–Ritz and Galerkin techniques have been previously used successfully to find such behaviour in the subcritical parameter range (Whiting, 1997; Wadee et al., 1997). An important development reported here is the application of the latter to a parabolic PDE

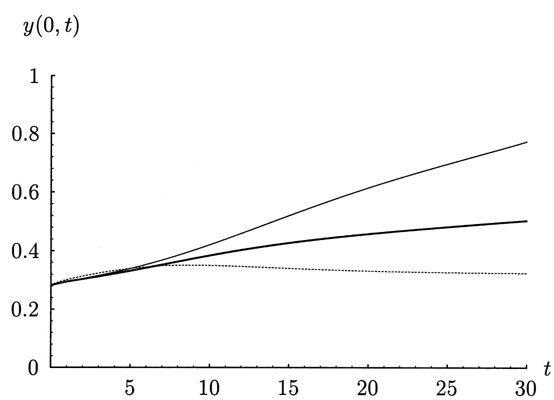


Fig. 11. Graph of central deflection vs time for (12). Thick line: numerical solution. Thin line: solution with method G2. Dashed line: solution with method G1.

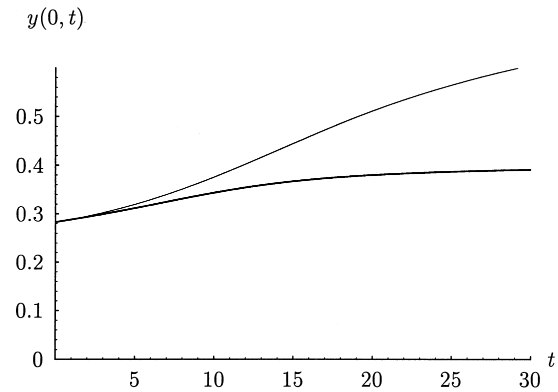


Fig. 12. $y(0, t)$ vs t for the linear PDE (15). Thick line: numerical solution. Thin line: solution using G2.

describing the behaviour of a thin elastic layer resting in a visco-elastic medium modelling a geological folding process (Hunt et al., 1996). The extra independent time variable causes difficulty because it means that the system is no longer conservative and so the hamiltonian nature of the elastic problem is not preserved and the deflection pattern of the elastic layer is history-dependent. When the layer is long, the boundary conditions play a small rôle in determining the overall profile but the loading conditions have profound effects. In the case studied here, we have dealt solely with *rigid* loading i.e. the end-shortening, $\mathscr{E}$, has been held constant for all t.

The initial triggering mode has been chosen to be localized because the initiation of the folding of the layer is assumed to be effectively instantaneous (in terms of a geological time scale) and, perhaps not surprisingly, the pattern continues to be localized for all time given that $\mathscr{E} = \text{constant}$. However, there is evidence that even a periodic initial profile eventually leads to localization (Budd and Peletier, 1999).

The Galerkin method has been extended in a way that the assumed solution need not be linear superpositions of modes. When applied to the study of the ODE, the method proves to be very good when compared against numerical solutions. In the study of the parabolic PDE, the results are less accurate far into the time evolution of the buckle pattern but are nevertheless clearly better than previous studies particularly in comparison with the standard Galerkin technique. Whereas in the elastic problem an adequate approximation to the solution can be reached through the properties of the

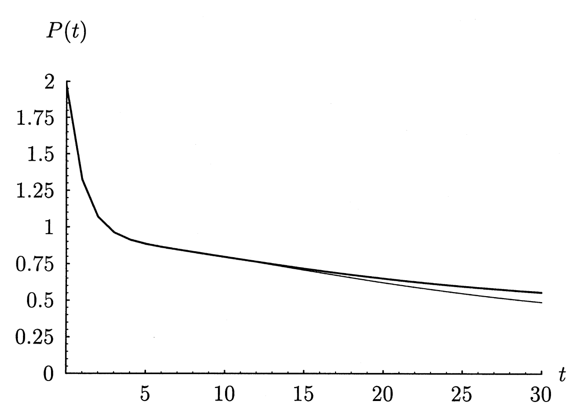
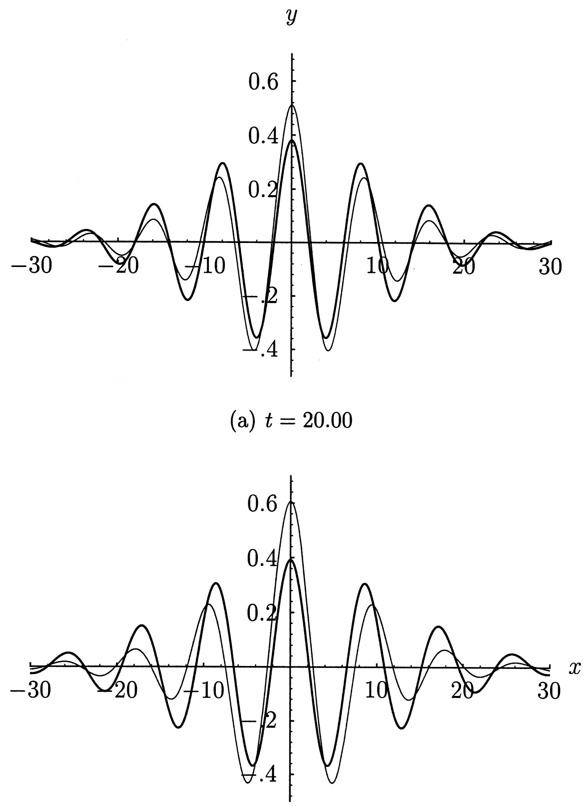


Fig. 13. P vs t for the linear PDE (15). Thick line: numerical solution. Thin line: solution using G2.



(a) $t = 20.00$

(b) $t = 30.00$

Fig. 14. Evolution of a localized deflection for $P(0) = 1.97$ for (15). Thick line: numerical solution. Thin line: solution using G2.

eigenvalues of the linearized governing equation (Whiting, 1997), in the visco-elastic problem, information about the shape factors is not available and a weighted residual method has been developed where such parameters can be treated as variables too.

In the current systems we were able to compare solutions with a collocation method which provides a useful validation procedure. Our aim is to proceed to analyse systems which are more closely related to geological folding where the assumption of a Winkler foundation in the model is at best dubious. Shear interaction is known to occur in the bedding of strata and so should be included. However, this adds an integral constraint on the system which cannot be solved using collocation. The modified Galerkin procedure provides a possible avenue for analysing the half space problem.

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